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Whittaker newforms for archimedean representations of GL(2)

Alexandru A. Popa

University of Pennsylvania, Philadelphia, PA, USA

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Abstract

We present a unified characterization of Whittaker newforms for infinite-dimensional admissible representations π of $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{GL}(2,\mathbb{C})$. We connect this problem with that of finding test vectors for toric linear forms on the representation space of π. This method allows us to treat in the same way the real and complex fields, thus identifying Whittaker newforms in all cases, including those previously unavailable in the literature.

1. Introduction

A Whittaker newform for a representation π of $G = \mathrm{GL}(2)$ over a local field F is a function in the Whittaker model whose Mellin transform equals the L-function of π. In the nonarchimedean case, such a newform can be found in the one-dimensional subspace of π invariant under a compact subgroup depending on the conductor of π. Here we provide a similar simple description of Whittaker newforms in the archimedean case as well, where Whittaker newforms have been previously computed for some, but not for all representations. Our approach follows a suggestion of B.H. Gross [Gr01, §11–12], and it is inspired by a connection between this problem, and that of determining test vectors for toric linear forms on the representation space of π. The connection with test vectors for linear forms allows us to treat the real and complex fields in a unified manner, thus identifying Whittaker newforms for all representations.

E-mail address: aapopa@gmail.com.

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Determining Whittaker newforms is important in the theory of automorphic representations, for example in the computation of Rankin–Selberg integrals over $\mathbb{R}$, e.g. see [Zh01]. In most real cases, this has been done in [JL70,Go70] or [Ge75], but in the complex case, Whittaker newforms have not been previously computed.

Let F be either $\mathbb{R}$ or $\mathbb{C}$, and let π be an admissible, irreducible, infinite-dimensional representation of $G(F)$, that is a $(\mathfrak{g}, K)$-module, where $\mathfrak{g}$ is the complexification of the Lie algebra of $\mathrm{GL}_2(F)$ and K is a maximal compact subgroup [Bu97, p. 200]. Let $W(\pi, \psi)$ be the Whittaker model of π with respect to a nontrivial additive character ψ of F. For any $W \in W(\pi, \psi)$, define the "Mellin transform":

$$\Psi_W(s, g) = \int_{F^\times} W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) |x|_F^{s-1/2}\, d^\times x \tag{1}$$

where $d^\times x$ is the multiplicative Lebesgue measure on $\mathbb{R}^\times$ or $\mathbb{C}^\times$, and the absolute value is the usual one on $\mathbb{R}$ and the square of the one on $\mathbb{C}$. The integral converges if the real part of s is large enough, and it can be meromorphically continued to the complex plane. The L-function $L(s, \pi)$ is defined as the greatest common denominator of all $\Psi_W(s, g)$, appropriately normalized. The choice of exponent $s - 1/2$ in formula (1) is made so that the Mellin transform has a functional equation for $s \to 1 - s$.

Our goal is to identify explicit vectors, called *Whittaker newforms* $W_\pi \in W(\pi, \psi)$ such that:

$$\Psi_{W_\pi}(s, e) = L(s, \pi), \tag{2}$$

where e is the unit matrix in $G(F)$. We will look for Whittaker newforms in the K-types of π (see Section 3 for the definition), and show that they can always be found in the minimal *test space* (see Definition 1), a one-dimensional subspace of the minimal K-type containing vectors W for which $\Psi_W(s, e)$ does not vanish. The proof is by direct computation: a Whittaker vector in the *minimal* test space can be explicitly computed in terms of Bessel functions, since it satisfies differential equations expressing the action of $\mathfrak{g}$ [Go70,JL70], and the result follows from integral identities satisfied by Bessel functions.

We emphasize the connection with test vectors for linear forms since it gives a conceptual framework that may prove useful for extending the notion of Whittaker newforms to higher rank groups. In the next section, we describe this connection and reformulate the problem in terms of finding test vectors for toric linear forms.

2. Test vectors for linear forms

In the next few paragraphs only, we let F be a local field, not necessarily archimedean. Let $\pi : G(F) \to \mathrm{Aut}\, V$ be an admissible, irreducible, infinite-dimensional representation with central character ω. Let T be the split two-dimensional torus over F and we identify $T(F)$ with the diagonal matrices inside $G(F)$. Consider a character $\chi : T(F) \to \mathrm{Aut}\, \mathbb{C}$ whose restriction to the center of $G(F)$ equals ω^{-1}.

Let $\mathrm{Hom}_T(\pi \otimes \chi, \mathbb{C})$ be the space of linear forms on V on which $T(F)$ acts by χ^{-1}. A special case of a theorem of J.-L. Waldspurger and J. Tunnell, shows that in this case the space $\mathrm{Hom}_T(\pi \otimes \chi, \mathbb{C})$ is one-dimensional (it is at most one-dimensional for any embedded torus T). A natural question is then of finding test vectors on which such a form is nonzero, and this problem has been treated in the nonarchimedean case by B.H. Gross and D. Prasad [GP91].

Let $l : V \to \mathbb{C}$ be the Whittaker functional, the unique linear form (up to a constant) on which the group of unipotent matrices acts by the character ψ. The connection between the test vector problem and that of finding Whittaker newforms arises from observing that the integral

$$m_\chi(v) = \int_{F^\times} l\big[\pi\big(a(t)\big)v\big]\chi\big[a(t)\big]\,d^\times t, \tag{3}$$

defines, when it converges, a linear form $m_\chi \in \mathrm{Hom}_T(\pi \otimes \chi, \mathbb{C})$. Here $a(t)$ is the diagonal matrix with upper entry t and lower entry 1. Taking $\chi = \chi_s$ with

$$\chi_s\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} := \left|\frac{a}{d}\right|_F^{s-\frac{1}{2}} \omega^{-1}(d), \tag{4}$$

the integral $m_\chi(v)$ becomes the Mellin transform denoted by $\Psi_{W_v}(s, e)$ before, where $W_v \in W(\pi, \psi)$ is given by $W_v(g) := l(gv)$ for $v \in V$. Therefore what we have called Whittaker newforms before provide test vectors for the toric linear forms associated with the character χ_s of (4).

In the nonarchimedean case, Proposition 2.6 in [GP91] implies that the linear form m_χ, when χ is unramified, does not vanish on the minimal nonzero subspace of V fixed by a compact subgroup of $G(\mathcal{O}_F)$ consisting of matrices congruent to the identity matrix modulo ϖ_F^n for some integer $n \geqslant 0$, where $\mathcal{O}_F$ is the ring of integers and ϖ_F is a uniformizer in F. The minimal subspace is one-dimensional, and the minimal n equals the conductor of π [Ca73]. For v_π an appropriately normalized vector in this one-dimensional space, it is well know that $m_\chi(v_\pi)$ equals the L-function of π, so the corresponding $W_\pi \in W(\pi, \psi)$ is a Whittaker newform.

We show that in the archimedean case the situation is entirely similar. The representation π, restricted to the standard maximal compact subgroup K of G, decomposes into a sum of finite-dimensional representations, called K-types. Inside each fixed K-type $\mathcal{W}$ we consider the *test space* of vectors $\mathcal{W}^T$ on which the compact torus $T_c = T \cap K$ acts by the character χ^{-1}. We show in Proposition 1 that $\mathcal{W}^T$ is at most one-dimensional, and when it is trivial, the linear form m_χ vanishes on the whole K-type $\mathcal{W}$. When $\chi = \chi_s$, the Whittaker newform is then found to reside in $\mathcal{W}^T$, for the "minimal" K-type $\mathcal{W}$ for which $\mathcal{W}^T$ is one-dimensional. This proves also that the corresponding linear form m_{χ_s} is nonzero on the minimal test space $\mathcal{W}^T$.

3. *K*-types

Henceforth we assume F is $\mathbb{R}$ or $\mathbb{C}$. We fix the characters $\psi(x) = e^{2\pi i x}$ of $\mathbb{R}$ and $\psi(z) = e^{2\pi i(z+\bar{z})}$ of $\mathbb{C}$, and we denote by K the standard maximal compact subgroup of $G = \mathrm{GL}_2(F)$, so $K = O_2(\mathbb{R})$ or $K = U_2(\mathbb{C})$, respectively.

The representation π restricted to K decomposes into a direct sum of finite-dimensional nontrivial subspaces, called *K-types*:

$$\pi|_K = \mathcal{W}_n \oplus \mathcal{W}_{n+2} \oplus \mathcal{W}_{n+4} \oplus \cdots,$$

indexed by the infinite set of nonnegative integers $\{n + 2k\}_{k \in \mathbb{Z}_{\geqslant 0}}$, to be described below. The smallest one, n, is called *the weight of* π.

For $F=\mathbb{R}$, the K-type $\mathcal{W}_m$ is the span of the vectors in the representation space of π having weight $\pm m$ under the action of $\mathrm{SO}_2(\mathbb{R})$. The vector v has weight m if

$$\pi(k_\theta)v=e^{im\theta}v \quad \text{for all } k_\theta=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\in \mathrm{SO}_2(\mathbb{R}).$$

The space $\mathcal{W}_m$ is 1-dimensional for $m=0$ and 2-dimensional for $m>0$. If π is a principal series representation with central character $\mu(t)=|t|^r\operatorname{sgn}(t)^n$ with $n\in\{0,1\}$, then the weight of π is n; if π is a discrete series representation $\sigma(\mu_1,\mu_2)$ with $\mu_1\mu_2^{-1}(t)=t^p\operatorname{sgn}(t)$ for some integer $p>0$, then the weight of π is $p+1$.

For $F=\mathbb{C}$, the K-type $\mathcal{W}_m$ is the $(m+1)$-dimensional subspace of π on which $SU_2(\mathbb{C})$ acts by its unique irreducible $(m+1)$-dimensional representation ρ_m. We recall that the representation ρ_m can be realized on the space V_m of degree m homogeneous polynomials in two variables:

$$\varphi(x,y)=\sum_{k=-m/2}^{m/2}a_k x^{m/2+k}y^{m/2-k},$$

on which $SU_2(\mathbb{C})$ acts by

$$\rho_m(g)\varphi(x,y)=\varphi\bigl[(x,y)g\bigr]. \tag{5}$$

If π is the principal series representation $\pi(\mu_1,\mu_2)$, with $\mu_1\mu_2^{-1}(z)=z^p\bar{z}^q$ such that $p-q$ is an integer, then the weight of π is $|p-q|$. Recall that all irreducible, admissible representations of $G(\mathbb{C})$ are isomorphic to a principal series representation.

4. Test space

Towards our goal of finding a Whittaker vector W_π whose Mellin transform equals the L-function, we first identify inside each K-type of π a natural "test space" for the nonvanishing of any linear form $m\in\operatorname{Hom}_T(\pi\otimes\chi,\mathbb{C})$. Recall that χ is a character of T whose restriction to the center equals ω^{-1}, hence we can write:

$$\chi\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}=\chi_1\bigl(ad^{-1}\bigr)\omega^{-1}(d),$$

with χ_1 a character of $F^\times$.

Definition 1. If $\mathcal{W}$ is a K-type of π, the *test space* associated to $\mathcal{W}$, denoted by $\mathcal{W}^T$ is the (possibly zero) subspace of $\mathcal{W}$ on which the compact torus $T_c:=T\cap K$ acts by χ^{-1}.

Explicitly, for $F=\mathbb{R}$

$$\mathcal{W}^T=\bigl\{v\in\mathcal{W}\colon\ \pi(\epsilon)v=\chi_1(-1)^{-1}v\bigr\}, \tag{6}$$

where $\epsilon=\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)\in T_c$, while for $F=\mathbb{C}$

$$\mathcal{W}^T = \{v \in \mathcal{W}\colon\ \pi[t(a)]v = \omega(a)^{-1}\chi_1(a)^{-2}v,\ \forall a \in S^1\},\tag{7}$$

where $S^1 = \{a \in \mathbb{C}\colon\ |a| = 1\}$, and $t(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a} \end{smallmatrix}\right) \in T_c$ for $a \in S^1$.

The following proposition justifies the definition, connecting the space $\mathcal{W}^T$ with the question of vanishing of linear forms on which T acts by χ^{-1}.

Proposition 1.

(i) *If $\mathcal{W}$ is an arbitrary K-type of π, then the space $\mathcal{W}^T$ is at most one-dimensional.*
(ii) *If $\mathcal{W}$ is a K-type of π such that* $\dim \mathcal{W}^T = 0$, *then* $m(v) = 0$ *for every linear form* $m \in \mathrm{Hom}_T(\pi \otimes \chi, \mathbb{C})$ *and for every* $v \in \mathcal{W}$.

Proof. *Case* 1: $F = \mathbb{R}$. If $\mathcal{W}$ is one-dimensional (i.e. π has weight 0 and $\mathcal{W}$ is the minimal K-type), then it is obvious that $\mathcal{W}^T$ is 0 or 1-dimensional, depending on the eigenvalue of ϵ on $\mathcal{W}$, hence $\dim \mathcal{W}^T \leqslant 1$ [see Eq. (6)]. The second part is obvious in this case.

On the other hand, if $\mathcal{W}$ is two-dimensional, the action of ϵ on $\mathcal{W}$ decomposes into two eigenspaces with eigenvalues ± 1, hence $\dim \mathcal{W}^T = 1$. There is nothing to prove for part (ii) in this case.

Case 2: $F = \mathbb{C}$. Let $\mathcal{W}$ be the (unique) $(n+1)$-dimensional subspace of π on which $SU_2(\mathbb{C})$ acts by its irreducible representation ρ_n.

Part (i) follows from the fact that the restriction of ρ_n to T_c decomposes completely into a direct sum of one-dimensional representations, each appearing with multiplicity one. If the character χ restricted to T_c is not among them, then $\dim \mathcal{W}^T = 0$; otherwise, $\dim \mathcal{W}^T = 1$.

For part (ii), notice that if $0 \neq v \in \mathcal{W}$ and if $m \in \mathrm{Hom}_T(\pi \otimes \chi, \mathbb{C})$, then by definition we have

$$m[\pi(t(a))v] = \omega^{-1}(a)\chi_1^{-2}(a)m(v) \quad \text{if } |a| = 1.$$

By (7), it follows that $m(v) = 0$ if T_c acts on v by a character different from $\omega^{-1}\chi_1^{-2}$. If $\mathcal{W}^T = 0$, we have observed above that $\mathcal{W}$ is spanned by such vectors v, hence m vanishes on $\mathcal{W}$. $\square$

5. Whittaker newforms

The previous proposition suggests that the space $\mathcal{W}^T$, when nonzero, provides test vectors on which linear forms $m \in \mathrm{Hom}_T(\pi \otimes \chi, \mathbb{C})$ do not vanish. To show that this is indeed the case, one can use the specific form $m_\chi \in \mathrm{Hom}_T(\pi \otimes \chi, \mathbb{C})$, defined on Whittaker functions by averaging as in (3). However, in this paper we are mostly interested in identifying Whittaker newforms, hence we restrict ourselves from now on to the case $\chi = \chi_s$, the character in Eq. (4). The linear form $m_\chi(W)$ becomes the Mellin transform $\Psi_W(s, e)$ for this choice of χ, and we have the following main theorem.

Theorem 1. *In both the real and the complex case, let $\mathcal{W}$ be the* minimal *K-type in the Whittaker model $W(\pi, \psi)$ such that* $\dim \mathcal{W}^T = 1$. *Then*

$$\Psi_W(s, e) = L(s, \pi)$$

for some $W \in \mathcal{W}^T$, and for $\mathrm{Re}(s)$ *large enough.*

We first describe explicitly the minimal K-type $\mathcal{W}$ of the proposition. In the course of the proof, we also compute explicitly the Whittaker newforms belonging to the corresponding test space $\mathcal{W}^T$.

- $F=\mathbb{R}$: Let π be a representation of $G(\mathbb{R})$ of weight n. The minimal K-type $\mathcal{W}$ for which $\mathcal{W}^T \neq 0$ is $\mathcal{W}_n$, unless π is the weight 0 representation $\pi(|\ |^{r_1}\operatorname{sgn}, |\ |^{r_2}\operatorname{sgn})$, when it is $\mathcal{W}_2$.
- $F=\mathbb{C}$: Let π be a principal series representation $\pi(\mu_1,\mu_2)$ of $G(\mathbb{C})$, where $\mu_i(z)=z^{p_i}\bar{z}^{q_i}$ are characters of $\mathbb{C}^\times$, with $p_i, q_i \in \mathbb{C}$ such that $p_i - q_i \in \mathbb{Z}$. If $n = |p_1 - q_1 - (p_2 - q_2)|$ is the weight of π, and $m = q_1 - p_1 + q_2 - p_2$ is the exponent of $\omega^{-1}(a)$ for $a \in S^1$, then the minimal K-type $\mathcal{W}$ for which $\mathcal{W}^T \neq 0$ is $\mathcal{W}_N$, with $N = \max(n, |m|)$.

Proof of Theorem 1. For $W \in W(\pi,\psi)$ and $t \in \mathbb{R}$, let $f_W(t)$ be the function of one variable:

$$f_W(t) = W\begin{pmatrix} |t|^{1/2}\operatorname{sgn} t & 0 \\ 0 & |t|^{-1/2}\end{pmatrix} \text{ or } W\begin{pmatrix} |t|^{1/2} & 0 \\ 0 & |t|^{-1/2}\end{pmatrix}, \tag{8}$$

depending on whether $F=\mathbb{R}$ or $\mathbb{C}$, respectively. Using the lowering and raising operators in the complexification of the Lie algebra of $\mathrm{GL}_2(F)$, we will show that if $W \in \mathcal{W}^T$, where W is the minimal K-type of the proposition, then f_W satisfies a second order differential equation satisfied also by the function $t^a J_u(4\pi t)$ for some $a, u \in \mathbb{C}$, where J_u is the Bessel function whose properties we recall below. Since f_W is of rapid decay at infinity, and the Bessel equation has only one solution of rapid decay, it follows that $f_W(t)$ is of the form $t^a J_u(4\pi t)$ (up to a constant). The theorem then follows from properties of the Bessel function which we recall below.

The Bessel function J_u satisfies the differential equation:

$$J_u''(y) + \frac{J_u'(y)}{y} - \left(1 + \frac{u^2}{y^2}\right) J_u(y) = 0 \quad \text{for } y > 0. \tag{9}$$

It can be shown that (up to a constant) this equation admits a unique solution of moderate growth at infinity. Normalized appropriately, this solution satisfies the following identities:

$$\int_0^\infty e^{-y(t+t^{-1})} t^u \, d^\times t = 2J_u(2y), \tag{10}$$

$$\int_0^\infty J_u(y) y^s \, d^\times y = 2^{s-2}\Gamma\left(\frac{s+u}{2}\right)\Gamma\left(\frac{s-u}{2}\right), \tag{11}$$

where $y > 0$ in the first equation, and $\operatorname{Re} s > |\operatorname{Re} u|$ in the second. We also need the equation satisfied by the functions $G(y) = y^a J_u(y)$, where $a \in \mathbb{C}$. It can be easily seen that

$$G''(y) + (1-2a)\frac{G'(y)}{y} - \left(1 + \frac{u^2 - a^2}{y^2}\right) G(y) = 0. \tag{12}$$

The real case. When π is a discrete series representation, a weight 1 principal series representation, or a weight 0 principal series representation of the type $\pi(|\ |^{r_1}, |\ |^{r_2})$, it is well known that the minimal weight Whittaker vector is a Whittaker newform, if appropriately normalized [JL70,Zh01]. The minimal weight vector belongs to $\mathcal{W}^T$ in these cases by the remark following the statement of the theorem. The only case left is when π is the weight 0 principal series $\pi(|\,|^{r_1}\operatorname{sgn}, |\,|^{r_2}\operatorname{sgn})$, when the minimal K-type of the proposition is $\mathcal{W}_2$.

Let $r = r_1 - r_2$. Let $W_2, W_0, W_{-2} \in \mathcal{W}_2$ be Whittaker vectors of weights 2, 0, −2, respectively, normalized (up to a constant) such that $L^2 W_2 = (r-1)LW_0 = (r^2-1)W_{-2}$, where $L \in \mathfrak{g}$ is the lowering operator. Then it is shown in [Go70, p. 2.7, Eq. (21)], that $\pi(\epsilon)W_2 = -W_{-2}$, hence $W_- := W_2 - W_{-2} \in \mathcal{W}_2^T$.

Let f_2, f_0, f_{-2} be the functions corresponding to W_2, W_0, W_{-2} by (8). The action of the raising and lowering operators translates into the following system of differential equations [JL70]:

$$\begin{gathered}
(r+1)f_2(t) = 2tf_0'(t) - 4\pi t f_0(t),\\
(r+1)f_{-2}(t) = 2tf_0'(t) + 4\pi t f_0(t),\\
f_0''(t) - \bigl[4\pi^2 + \bigl(r^2-1\bigr)/4t^2\bigr] f_0(t) = 0.
\end{gathered}$$

Subtracting the first two equations we obtain ($r = r_1 - r_2 \neq \pm 1$ because π is a principal series representation)

$$f_2(t) - f_{-2}(t) = -8\pi t f_0(t)/(r+1),$$

while the third equation has as solution $f_0(t) = -t^{1/2} J_{r/2}(2\pi t)(r+1)/8\pi$ (compare with Eq. (12)). For $W = W_- \in \mathcal{W}_2^T$ we have

$$\Psi_{W_-}(s,e) = \int_0^\infty t^{(r_1+r_2)/2} t^{3/2} J_{r/2}(2\pi t) t^{s-1/2}\, d^\times t = L(s,\pi)$$

where for the second equality we have used formula (11). The integral $\Psi_{W_+}(s,e)$ vanishes for $W_+ = W_2 + W_{-2}$, because the function f_+ corresponding to W_+ by (8) is an odd function. Therefore,

$$\Psi_{2W_2}(s,e) = \Psi_{W_-}(s,e) + \Psi_{W_+}(s,e) = L(s,\pi),$$

which proves that the Whittaker newform can be taken to be the weight two function $2W_2$.

The complex case. Since all irreducible, admissible representations of $\mathrm{GL}_2(\mathbb{C})$ are isomorphic to principal series representations, we can assume that π is of the type $\pi(\mu_1, \mu_2)$. Here $\mu_i(z) = z^{p_i}\bar{z}^{q_i}$ are characters of $\mathbb{C}^\times$, with $p_i, q_i \in \mathbb{C}$ such that $p_i - q_i \in \mathbb{Z}$.

Let $\mu(z) := \mu_1\mu_2^{-1}(z) = z^p\bar{z}^q$, and assume without loss of generality that $p \geqslant q$ (since $\pi(\mu_1,\mu_2) \simeq \pi(\mu_2,\mu_1)$). Then π has weight $n = p - q$, and the restriction of π to $SU_2(\mathbb{C})$ decomposes as follows:

$$\pi|_{SU_2(\mathbb{C})} \simeq \rho_n \oplus \rho_{n+2} \oplus \rho_{n+4} \oplus \cdots, \tag{13}$$

where ρ_N is the unique $(N+1)$-dimensional irreducible representation of $SU_2(\mathbb{C})$, acting as in (5).

Next we compute the Whittaker vectors belonging to the minimal K-type $\mathcal{W}$ of the proposition, using the differential equations satisfied by their diagonal functions [JL70]. Fix an integer $N \geqslant n$, having the same parity as n, so that ρ_N appears in the decomposition (13), and fix an $SU_2(\mathbb{C})$ intertwining map

$$i_N : V_N \to W(\pi, \psi). \tag{14}$$

For k between $-N/2$ and $N/2$, let $W_k \in W(\pi, \psi)$ be the image of the monomial $x^{N/2+k} y^{N/2-k}$. That is W_k is in the image of i_N and transforms as follows under T_c:

$$W_k\big[gt(a)\big] = a^{2k} W_k(g) \quad \text{for } |a| = 1.$$

It is clear that W_k is then completely determined by the function $f_k(t) := f_{W_k}(t)$ defined in (8), for $t > 0$. Using the action of certain elements in the center of the universal enveloping algebra of $\mathrm{GL}_2(\mathbb{C})$, it is shown in [JL70] that the functions $f_k(t)$ satisfy the following differential equations[1] for $-N/2 \leqslant k \leqslant N/2$:

$$f_k''(t) - (1-2k)\frac{f_k'}{t} - \left(16\pi^2 + \frac{p^2 - (1-k)^2}{t^2}\right) f_k(t) = -8\pi i (N/2 + k)\frac{f_{k-1}}{t}, \tag{15}$$

$$f_k''(t) - (1+2k)\frac{f_k'}{t} - \left(16\pi^2 + \frac{q^2 - (1+k)^2}{t^2}\right) f_k(t) = 8\pi i (N/2 - k)\frac{f_{k+1}}{t}. \tag{16}$$

Taking $k = -N/2$ in Eq. (15) and $k = N/2$ in Eq. (16) and comparing with Eq. (12) we see immediately that $f_{-N/2}$, $f_{N/2}$ are related to Bessel functions as follows (up to a constant which we choose to be one):

$$f_{-N/2}(t) = t^{1+N/2} J_p(4\pi t), \qquad f_{N/2}(t) = t^{1+N/2} J_q(4\pi t). \tag{17}$$

Moreover, if $N = n$ (the minimal K-type) one can compute all the functions $f_k(t)$, $-n/2 \leqslant k \leqslant n/2$, in terms of Bessel functions (which does not seem easily feasible for higher N). With our assumption that $n = p - q \geqslant 0$, one can show proceeding recursively that

$$f_k(t) = t^{n/2+1} J_{q+n/2-k}(4\pi t), \quad \text{for } k = n/2, n/2-1, \ldots, -n/2. \tag{18}$$

Coming back to the proof of the proposition, denote by m the integer such that $\omega^{-1}(a) = a^m$ if $|a| = 1$, that is $m = q_1 - p_1 + q_2 - p_2$. As pointed out following the statement of the proposition, the minimal K-type $\mathcal{W}$ such that $\mathcal{W}^T \neq 0$ is isomorphic to ρ_N, where

$$N = \max\big(n, |m|\big).$$

Given the definition of $\mathcal{W}^T$, for $W \in \mathcal{W}^T$ we have

$$\Psi_W(s, e) = \int\limits_{\mathbb{R}_{>0}} W\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} t^{2s-1}\, d^\times t. \tag{19}$$

[1] There is a factor of 4π missing in [JL70]. To correct the equations there, u has to be replaced by $4\pi u$.

To compute this integral, we consider two cases:

Case 1: $N = |m| > n$, that is $(p_1 - q_1)(p_2 - q_2) > 0$. Since we have assumed $p \geqslant q$, we must have $p_i > q_i$, hence $m = -N < 0$ and

$$L(s, \pi) = G_2(s + p_1)G_2(s + p_2).$$

On the other hand, the space $\mathcal{W}^T$ is spanned by $W_\pi = W_{-N/2}$, hence by Eq. (17):

$$W_\pi \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = \omega\big(t^{1/2}\big) W_\pi \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix} = t^{p_1+p_2+1} J_{p_1-p_2}(4\pi t).$$

Plugging this expression back into the formula (19) and using formula (11), we obtain the desired identity (2) (provided W_π is appropriately normalized).

Case 2: $N = n \geqslant |m|$, that is $(p_1 - q_1)(p_2 - q_2) \leqslant 0$. Since we have assumed $p \geqslant q$, we have also $p_1 \geqslant q_1$, $q_2 \geqslant p_2$, hence

$$L(s, \pi) = G_2(s + p_1)G_2(s + q_2).$$

In this case, the space $\mathcal{W}^T$ is spanned by $W_\pi = W_{m/2}$, hence by Eq. (18) we have (taking into account the relations among the p_i, q_i)

$$W_\pi \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = \omega\big(t^{1/2}\big) W_\pi \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix} = t^{p_1+q_2+1} J_{p_1-q_2}(4\pi t).$$

As before, identity (2) follows. □

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